Chapter 9 Chaos in non-dissipative systems*

Goals

- To understand the distinction between dissipative and nondissipative dynamical systems.
- To distinguish between integrable and nonintegrable systems.
- To distinguish between periodic and quasiperiodic behavior in integrable systems.
- To understand the relationship between the growth of mode-locking and the onset of chaos in weakly nonintegrable systems as embodied in the KAM theorem.

Comparing routes to chaos

Much of our work up to now dealt with one scenario for the onset of chaotic behavior. This was the period-doubling scenario, applicable to many iterative maps and to many physical systems such as the doublewell oscillator. In this chapter we deal with another scenario that occurs in many situations where period-doubling behavior is ruled out. Before exploring this scenario, we recall the features of the period-doubling behavior. We shall see that many of these features have analogs in the new scenario.

Period doubling is seen in the qualitative form of the Poincaré return map of the system after it has settled into its long-time behavior and reached an attractor. For the simplest motion, the map consisted of discrete points representing periodic behavior. The attractor thus occupied a vanishing fraction of the possible states of the system. As one increased a driving parameter, the number of such points increased in discrete steps. Beyond a well-defined threshold value of the parameter the number of points became infinite. Moreover, the points covered a nonzero fraction of the possible states of the system. This fraction increased from zero as the driving parameter increased above the threshold. Though the fraction was nonzero it was split up into an arbitrarily large number of discrete regions or bands. As the driving increased, the bands merged until ultimately only a single band remained.

Non-dissipative systems

Though a substantial class of mechanical systems show standard period-doubling behavior, there are several other types of behavior that can occur. In this section we sketch some of the important theorems governing this behavior. We now consider dynamical differential equations that have no dissipation and no driving. Thus the vector \( \vec{z} \) representing all the positions and momenta obeys an equation of the form \( \frac{d\vec{z}}{dt} = \vec{F}(\vec{z}) \). Since there is no driving, there is no explicit time dependence in \( \vec{F} \). Such systems are called autonomous.

Our rationale for period-doubling behavior from Chapter 7 cannot be applicable to these autonomous systems. An important ingredient for this rationale in eg. the doublewell oscillator was dissipation. There was a mechanism for the irreversible loss of energy: the oscillator was damped. Autonomous systems with dissipation simply slow down and stop, and thus they cannot display continuing chaotic behavior. Since the doublewell was dissipative, we could make an important observation about it: areas in phase space continually decrease with time. Thus motion is inevitably attracted to a one-dimensional attractor. The one-dimensional character enabled us to make a correspondence to one-dimensional iterative maps and thence to period-doubling behavior. Autonomous systems cannot follow this rationale, since they aren’t dissipative (or aren’t chaotic). The question arises: what happens instead?

two dimensions The simplest autonomous systems have a single co-ordinate and a single momentum, like the double well oscillator of the last chapter. They thus have a two-dimensional phase space. There are strong limits on how chaotic an autonomous, two-dimensional system can become. If the system is a mechanical system with a conserved energy, we have seen in Chapter 6 that the motion is integrable: one can express the position as a function of time in terms of explicit integrals. But even when the system is a more general two-dimensional system, there are analogous restrictions, codified in two theorems: the No-Crossing Theorem and the Poincaré-Bendixson Theorem. More details are available in the book by Hilborn on the reading list.

* latest updates indicated by footnotes
The no-crossing theorem observes that trajectories in phase space cannot cross. If two trajectories were to cross, they would have a common intersection point \( \bar{z} \). Then they would leave the intersection in different directions. However, if two different trajectories were to lead to the same phase space point \( \bar{z} \), these two would be bound to have the same \( d\bar{z}/dt \), since \( \bar{z} \) determines \( d\bar{z}/dt \). That means the two cannot go in different directions: intersection is impossible. In two-dimensions the no-crossing theorem has a dramatic consequence. If there is a closed orbit, all points that start inside the orbit must stay inside forever.

The Poincaré Bendixson theorem shows that two-dimensional motion is often forced to form closed orbits, i.e., repeating, periodic motion. In particular if one knows that a given motion remains bounded for all times, this motion must form a closed orbit. This theorem is proved by considering two trajectories adjacent to the one in question—one on either side. The trajectory in question is trapped between these two others. There is so little room for the trajectory to manoeuvre, that it is obliged to close on itself.

**nearly integrable motion** We now relax our restriction of two-dimensional phase space but add a new one about the type of motion. We consider mechanical motion with a potential energy function \( V(\bar{x}) \). The corresponding equations of motion for the co-ordinates \( x_a \) and their momenta \( p_a \) are \( dx_a/dt = p_a \); \( dp_a/dt = -\partial V/\partial x_a \). The coupled vibrations of a musical instrument is an example of this kind of dynamics. A group of asteroids orbiting each other is another. A collection of charged particles subject to an external static electric field is another example. In such a system, energy is conserved: it cannot change in time. Moreover, such a system preserves areas in phase space, unlike the double-well oscillator.

The no-crossing theorem observes that trajectories in phase space cannot cross. If two trajectories are of the same kind, they cannot go in different directions: intersection is impossible. In two-dimensions the no-crossing theorem has a dramatic consequence. If there is a closed orbit, all points that start inside the orbit must stay inside forever.

Further, motion in this system is **time reversible**: if \( x_a(t) \) is a possible motion, so is \( x_a(-t) \). Sometimes there are further conserved quantities as well. This system is a subclass of a more general class of autonomous Hamiltonian systems, all of which have a conserved Hamiltonian function, conservation of phase space area, and time-reversible motion. Though we will concentrate on potential-energy driven motion, the behavior we will describe is generally applicable to all Hamiltonian systems. Thus the discussion below describes instabilities of interacting particle beams in accelerators and confined plasmas in the laboratory or around stars and galaxies. This chapter provides only a sketch of the behavior of chaos in these systems. For a more complete picture, see Chapter 7, in Edward Ott *Chaos in Dynamical Systems* 2nd edition, (Cambridge, 2002).

Powerful things can be said about such systems when they are **nearly integrable**. In this new context, integrable means that new co-ordinates \( y_a(\bar{x}) \) can be found such that the motions for different \( a \) are independent. That is, \( \partial V/\partial y_a \) depends only on \( y_a \) itself and not on the other \( y \)'s. In the central force motion of Chapter 6, the \( y_1 \) is the radial co-ordinate and \( y_2 \) is the angular position around the center. (The \( \partial V/\partial y_a \) has to be taken with constant angular momentum.) Often one can deduce that a certain motion is integrable even though one doesn’t know the the magic co-ordinates \( y_a \) explicitly.

An integrable system is essentially a pair of independent two-dimensional systems. If the motion remains bounded, each of these must live in a periodic orbit with some repeat time \( \tau_a \). This means the overall motion must be either periodic or **quasi-periodic**. If all the periods \( \tau_a \) are integer multiples \( N_a \) of some \( \tau_0 \), then the whole motion repeats in a time which is \( N_0 \) times the product of all the \( N_a \) or smaller. This **periodic** motion occurs whenever the ratios of the all the \( \tau_a \)'s are rational. If some ratios are irrational, the motion is said to be **quasi-periodic**. In this case, the motion never repeats. If the periods \( \tau_a \) are chosen at random, the chance that they are rationally related is zero. However, there are rationally-related \( \tau_a \)'s that are arbitrarily close to the chosen ones. Whether the motion is periodic or quasiperiodic, the motion of any co-ordinate, say \( x_1 \) can be expressed as a sum of a finite set of periodic functions. The number of different functions needed is no more than the number of co-ordinates. This motion is not chaotic, since two initially adjacent points do not spread apart indefinitely; they come back together arbitrarily closely in a finite time

**KAM theorem** Now suppose that we have an integrable system whose potential is \( V_0(\bar{x}) \). Many physical systems are nearly integrable but not quite. These have potentials \( V(\bar{x}) \) that can be written in the form \( V(\bar{x}) = V_0(\bar{x}) + \epsilon V(\bar{x}) \), where \( V_0 \) gives rise to integrable motion and the parameter \( \epsilon \) is small. Figure 9.1 gives an example. The picture shows a set of pendula swinging at large amplitude, perhaps going over the top. Each pendulum by itself would execute periodic motion, with a period \( \tau \) that is longer for

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† modified 30Nov 2012

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bigger amplitude. The potential $V_0$ is that of the isolated pendula. The motion is obviously integrable, and the magic co-ordinates $y_\alpha$ are just the original pendulum angles. For convenience in what follows, we measure these angles in units of revolutions, so that $y+1$ and $y$ correspond to the same pendulum position. Now we add a weak perturbing potential $\epsilon V$ in the form of bar magnets on the pendula. We would like to know how the $\epsilon$ term alters the motion after a long time. This question is the subject of the famous theorem of Kolmogorov, Arnold and Moser known as the KAM Theorem. [see Ott, op. cit.] First we suppose that various amplitudes have been chosen so that the motion without the $\epsilon$ coupling is periodic. That is, for some integers $N_1$ and $N_2$ $N_1\tau_1 = N_2\tau_2 = \ldots \equiv \tau$. We say that the $\tau_\alpha$ are commensurate. We may label the co-ordinates in order of their periods, so that $\tau_1$ is the longest one. The motion repeats after every time $\tau$. This is true for any choice of initial angles $y_1$, $y_2$. We could delay any of the pendulum oscillators by an arbitrary time, and the others would be unaffected. In other words, the repeating motion has neutral stability with respect to these delays.

This neutral stability disappears as soon as we add a nonzero coupling $\epsilon$. Instead, the delays are fixed to a set of discrete values. To understand this effect, it is sufficient to consider just two pendula. To specify the delay we note the value of $y_2$ whenever $y_1$ is, say, 0. The coupling means that pendulum 1’s effect on pendulum 2 depends on their relative angles. To illustrate, we suppose that $\tau_1$ and $\tau_2$ are equal. Now suppose that the angle $y_1$ is ahead of $y_2$. The magnets tend to pull the two pendula towards each other. Thus $y_2$ will not exactly repeat after a time $\tau_2$; instead it will move ahead slightly. The neutral stability is lost. Only if $y_1 = y_2$ or $y_1 = y_2 + \frac{1}{2}$ will there be no change in $y_2$ over a cycle. In either case, there is no net influence to make $y_2$ advance or decrease over a cycle. If $\tau_2/\tau_1 = N_1/N_2$ is not 1, pendulum 1 will still cause a slight change in $y_2$ over the periodic time $\tau \equiv N_2\tau_2$. Some initial angles $y_2$’s will advance, over a $\tau$ cycle; others will decrease. In between there must be at least one $y_2$ that does neither. It is a fixed point. The image of this point after one $\tau_2$ cycle must also have no net effect after $\tau$: it too is a fixed point of the $\tau$ cycle. Repeating this argument, we infer that there are $N_2$ fixed points of the $\tau$ cycle. These $y_2$ values are points of an $N_2$-cycle. Thus for every point in the unperturbed system where $\tau_2/\tau_1$ is rational, we must have a set of special angles $y_2$ where the motion is periodic, but for general choice of $y_2$ it is not. In general there are not just $N_2$ but $2N_2$ choices of the fixed-point angle, just as there were two fixed $y_2$’s for the case of equal periods. The structure of these fixed points regions is described by the Poincaré-Birkhoff Theorem. We shall deal with the stability of these cycles shortly.

In general the $\tau$’s of our system are not commensurate. As noted above, if we alter our initial state slightly, the periods $\tau_\alpha$ shift and the their ratios in general become irrational. The motion does not repeat in any finite time $\tau$. If we record $y_2$ when $y_1 = 0$ we get a uniform coverage of $y_2$ values over time. Now when we add a slight coupling $\epsilon$ there is no cumulative effect that would increase or decrease $y_2$ again and again over each period $\tau$. Any effect of $y_1$ on $y_2$ is transitory and averages out over time. Because of this, there are no fixed points $y_2$ for arbitrarily small $\epsilon$. The initially quasiperiodic orbit persists even for $\epsilon > 0$. However,
as $\epsilon$ increases, the time $t_\epsilon$ required for $y_1$ to affect $y_2$ decreases. If $\tau_2/\tau_1$ is very close to a rational value, then the motion may be indistinguishable from a periodic motion over the limited time $t_\epsilon$. Then we expect this coupling $\epsilon$ to create fixed-point angles $y_2$ as in the commensurate case above. As $\epsilon$ increases, a wider and wider range around each rational $\tau_2/\tau_1$ becomes subject to these fixed points. In general there are thus bands of $\tau_2/\tau_1$ where $y_2$ behaves incommensurately, separated by bands containing fixed points at specific $y_2$'s. For sufficiently large $\epsilon$ all the incommensurate bands separating regions of different $p$ may disappear. The KAM Theorem tells the relative robustness of each irrational ratio according a specific criterion of closeness to rationals. The harder the ratio is to approximate by rationals, the larger the $\epsilon$ value needed to create fixed points. There is one ratio that is hardest to approximate by rationals. This one requires the largest $\epsilon$ before the motion is subject to fixed points. This ultimate ratio is the Golden Mean encountered in Chapter 2. If $\epsilon$ remains smaller than this limit, there are some irrational orbits. These KAM orbits are uncrossable because they extend to form a closed surface in the space of $y_1, y_2$ values. Thus any point that starts within a KAM orbit must stay within it forever. The KAM orbits divide the phase space into non-communicating regions. To completely specify the KAM orbits we must give the momenta $p_1$ and $p_2$ for each point $y_1, y_2$. These points form a two-dimensional surface in the phase space $y_1, p_1, y_2, p_2$. Since every $y_1$ is the same point as $y_1 + 1$ and similarly for $y_2$, this surface has the topology of a torus. Thus we may say that the motion of any initial point is confined within the KAM tori surrounding that point.

**simplest embodiment of KAM chaos** We can learn a lot about the generic behavior of such systems by considering an especially simple system, called the *Standard Map*. The standard map is a Poincaré section for a physical system called the kicked rotor [Ott op. cit.]. The kicked rotor is a single physical pendulum without gravity that periodically receives a vertical, impulsive force of magnitude $\epsilon^\dagger$. As above, the angular position $x$ of the rotor will be measured in revolutions, so that $x+1$ is the same angular position as $x$. It has no dissipation and its motion preserves phase space area. The angle and momentum just before kick $i$ are denoted $x_i, p_i$. Because only the tangential component of the vertical kick affects the momentum $p$, this $p$ gets increased by $\epsilon \sin(2\pi x_i)$. It remains fixed until the next kick, so just before the next kick, $p_{i+1} = p_i + \epsilon \sin(2\pi x_i)$. The $x$ variable increases at a rate proportional to the momentum $p_{i+1}$. Thus just before the next kick $x_{i+1} = x_i + p_{i+1}$. Summarizing,

$$p_{i+1} = p_i + \epsilon \sin 2\pi x_i; \quad x_{i+1} = (x_i + p_{i+1}) \mod 1 \quad (9.1)$$

This Standard Map plays a role analogous to the Logistic Map in understanding non-dissipative Hamiltonian systems. The Poincaré section plotting the $(x_i, p_i)$'s visited after many iterations is shown in Figure 9.4 ‡.

For $\epsilon = 0$ the kicked rotor is integrable. The co-ordinate $p$ remains fixed and $x$ increases in fixed increments. If $p$ is rational, the motion is periodic; if it is irrational, the motion is quasiperiodic. As $\epsilon$ increases from 0, things become more complicated. For small $\epsilon$ the plane divides into many closed lines, some threading horizontally across the plot, others forming concentric loops (Figure 9.4). These lines gradually merge into regions of nonzero area as $\epsilon$ increases. The last stable quasi-periodic orbit occurs at $\epsilon \simeq 0.95/2\pi$. [J. M. Greene J. Math. Phys. 20 1183 (1979)]

As with the double pendulum, the kicked rotor with $\epsilon \neq 0$ has fixed points. We now consider the stability of these points, taking the point $p = 1/2$, $x = 1/2$ as an example. We note that $\sin(2\pi x) = \sin(\pi) = 0$. Thus after one iteration we have $p_1 = 1/2$, $x_1 = 1$ (or 0). After two iterations we have $p_2 = 1/2$, $x_2 = 1/2$. Thus $(x, p)$ is a point of a fixed two-cycle for any $\epsilon$.

To probe the stability of this fixed point, we consider a small displacement with $x = 1/2 + \delta$, $p = 1/2$. Then after one cycle, expanding the sin, we have, $p_1 = \frac{1}{2} \pi \epsilon \delta$, and $x_1 = \delta - \pi \epsilon \delta$. After two cycles, $p_2 = 1/2 - \pi \epsilon \delta + \epsilon (2\pi \epsilon \delta)$, and $x_2 = ((1/2 + \delta) - \pi \epsilon \delta - \pi \epsilon \delta + 2\pi \epsilon^2 \delta)$. We see that $x$ moves back towards the fixed point, while $p_2$ moves away to smaller values. If we had displaced $p$ downward and kept $x = 1/2$, we would see that $p$ moves back towards the fixed point while $x$ decreases. Evidently $(x_1, p_1) = (0, 1/2)$ behaves identically. The stability of this fixed point is not clear. However we may note that if $\epsilon$ is negative, both $x$ and $p$ move away from the fixed point: it is unstable.

† Our double pendula have a dynamics similar to the kicked rotor if pendulum 1 is made infinitely massive, so that pendulum 2 has negligible effect on pendulum 1.

‡ added 30Nov2012
To analyze the stability further, we would have to compute the stability matrix defined in Chapter 7. This is the matrix of partial derivatives of \( p_{i+1}, x_{i+1} \) with respect to \( p_i \) and \( x_i \). Then as in Chapter 7 we can determine its eigenvalues \( \lambda_+ \) and \( \lambda_- \). For this conservative system, these \( \lambda \)'s are tightly constrained by the equal-area theorem: \( |\lambda_+ \lambda_-| = 1 \). This may be satisfied in two ways. First, the two \( \lambda \)'s can be real with \( |\lambda_+| > 1 \) and \( |\lambda_-| < 1 \). In that case the fixed point is generically unstable with one single direction where the motion approaches it, as discussed in Chapter 7. The second alternative is for the \( \lambda \)'s to be complex conjugates, with equal magnitudes. In this case the fixed point is *neutrally* stable: the motion is a closed orbit around the fixed point neither converging nor diverging over time. The orbital period is clearly very long if \( \epsilon \) is small. For properly-chosen \( \epsilon \) the orbit can be commensurate. Otherwise it is quasiperiodic, covering the orbit uniformly as time proceeds.

The point \( x = p = 1/2 \) analyzed above is of this orbiting type for positive \( \epsilon \) and is unstable for negative \( \epsilon \). Since \( \sin(\theta + \pi/2) = -\cos(\theta) \), the unstable fixed point for negative \( \epsilon \) must correspond to an unstable fixed point at \( x = 1/4 \) and positive epsilon. Thus we have two orbiting fixed points and two unstable ones, as announced above. For other rational momenta, we may also find fixed points of the Standard Map. In general \( N \) cycles are needed to return to the fixed point. The \( N \) kick contributions to the momentum must sum to zero: \( 0 = \sum_i \sin(2\pi(x + i/N)) \). This more complicated condition for \( x \) means \( x = 1/2 \) is not always a fixed point.

How does chaos emerge in the Standard Map? As with the Logistic Map, the long time motion moves from orbits of measure zero in the phase space to regions or bands, densely covering a nonzero fraction of the phase space. Unlike the Logistic Map, the chaotic regions are in general disjoint and don’t communicate with each other. Further, there is a bit of chaos always present for arbitrarily small coupling parameter \( \epsilon \). We have seen above that there are orbits that lead to unstable fixed points. Two points initially straddling such an orbit must move far apart even if they are initially close together. However almost all orbits are quasiperiodic and have no fixed points and no chaos. The chaotic regions grow as these quasiperiodic orbits collapse with the growth of \( \epsilon \).

The Standard Map can readily be implemented in java. We may explore the development of periodic orbits and chaotic regions by placing a few points on the Poincaré section at random. The java program listed below follows six such points, shown in different colors. One can select the number of iterations, vary the coupling \( r \) or reset the initial points. One may also choose the red momentum explicitly. It is initially set to the inverse of the Golden Mean. The Java program is included in the same directory as this chapter.

```java
import java.awt.
import java.awt.event.*;
import java.lang.*;
import javax.swing.*;
import P251.*;
import java.awt.Color;
public class StandardMap extends P251Applet {
    final double pmax=1., xmax=1.;
    double r= 0.020, pRED = 0.; int nstep=0, n=100;
    Point[] z = new Point[6];
}
```
final static Color[] COLORS = {Color.RED, Color.ORANGE, Color.darkGray, Color.GREEN, Color.BLUE, Color.MAGENTA};
drawPanel2 dp;
inputPanel ip;
public void fillPanels() // specifies and outputs the drawing panel object to the applet
{
    dp = new drawPanel2(900,700); // creates the drawPanel object that will contain the drawing information
    dp.setDrawBounds((float)0., (float)-.5, (float)1., (float)1.);
    addPanel(dp);
    ip = new inputPanel();
    ip.addField("red momentum =", pRED);
    ip.addField("coupling r", r);
    ip.addField("# of steps", n);
    ip.addField("now at ", nstep);
    addPanel(ip);
} // end of fillPanels
public void initValues() { //initializes x, p points of the trajectories to be followed
    z[0] = new Point(.5, 2./(1. + Math.sqrt(5.))); //red trajectory has golden mean momentum
    ip.setValue(0, z[0].p);
    for (int i=1; i<z.length; i++) {z[i] = new Point(Math.random(), Math.random());} //initialize other trajectories
    nstep = 0;
} //end of initValues
public void readValues() { // read red momentum, r and n from input fields
    z[0].p = ip.getValue(0);
    r = ip.getValue(1);
    n = (int)ip.getValue(2);
} //end of readValues
public void compute() {
    readValues();
    int nlast=nstep+n;
    while(nstep<nlast) {
        nstep++;
        for (int i=0; i<z.length; i++) {
            z[i].iterate(1);
            dp.addLine((float)z[i].x, (float)z[i].p,(float)z[i].x, (float)z[i].p+.001, COLORS[i]);
        }
        if (Thread.interrupted()) { return; }
        if (((nlast - nstep) % 500) == 0) { //update display every 500 steps
            ip.setValue(3, nstep);
            ip.setValue(0, z[0].p);
            putmarks();
            dp.repaint();
            try{Thread.sleep(300);}
            catch(InterruptedException e) {
                System.out.println("there was an error sleeping");
                return;
            } //end of catch
        } //end of if for updating
    }
} // end of compute

void putmarks() { //marks rational values of p on the right side of the screen.
    final float X0 = (float).1;
    dp.addLine((float)1., (float)0. , (float)(1.-X0) ,(float)0.);
    dp.addLine((float)1., (float)1. , (float)(1.-X0) ,(float)1.);
    dp.addLine((float)1., (float).5, (float)(1.-.5*X0) , (float).5);
    dp.addLine((float)1., (float).25, (float)(1.-.25*X0) ,(float).25);
    dp.addLine((float)1., (float).75, (float)(1.-.25*X0) , (float).75);
    dp.addLine((float)1., (float).333, (float)(1.-.333*X0) , (float).333);
    dp.addLine((float)1., (float).667, (float)(1.-.333*X0) , (float).667);
    dp.addLine((float)1., (float).2, (float)(1.-.2* X0) , (float).2);
    dp.addLine((float)1., (float).4, (float)(1.-.2*X0) , (float).4);
    dp.addLine((float)1., (float).6, (float)(1.-.2*X0) , (float).6);
    dp.addLine((float)1., (float).8, (float)(1.-.2*X0) , (float).8);
}

private class Point extends Object { //single x, p point.
    public double x, p;
    Point() {x = 0.; p = 0.;}
    Point(double xx, double pp) {x = xx; p = pp;}
    public void iterate(int maxiter) { //iterates x, p according to Standard Map.
        for (int iter= 1; iter<=maxiter; iter++) {
            p = p + r *Math.sin(2.*Math.PI * x);
            x = x + p;
            while(x > 1.) {x--;} //assures that 0 < x < 1
            while(x < 0.) {x++;}
        } //end of iter loop
    } //end of iterate
}

Figure 9.3. Java program to plot attractors of the standard map.

Examples of the long-time behavior are shown in Figure 9.4. Many nearly quasiperiodic regions show up as wavy lines that are periodic from left to right. Other quasiperiodic regions are orbits around fixed points. Chaos cannot penetrate into the quasi-periodic orbits. Thus the chaotic bands are separated from the regions of discrete orbits.
Figure 9.4 Fifty thousand iterates of the Standard Map of Eq. (9.1) with $\epsilon = .02$ (left) and .15 (right). Horizontal axis is $x$; vertical axis is $p$. Five points in the $x$–$p$ plane were selected at random, the red points started with $x = 1/2$ and a momentum $p$ equal to the inverse of the Golden Mean. The marks on the right indicate where $p = 0, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5$ and 1.